NORTH-HOLLAND

On the Invariance of Colin de Verdière's Graph Parameter Under Clique Sums

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ABSTRACT

For any undirected graph G, let $\mu(G)$ be the graph invariant introduced by Colin de Verdière. In this paper we study the behavior of $\mu(G)$ under clique sums of

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graphs. In particular, we give a forbidden minor characterization of those clique sums G of G_1 and G_2 for which $\mu(G) = \max\{\mu(G_1), \mu(G_2)\}$.

1. INTRODUCTION

Colin de Verdière [2] (cf. [3]) introduced an interesting new invariant $\mu(G)$ for graphs G, based on algebraic and analytic properties of matrices associated with G. He showed that the invariant is monotone under taking minors and that $\mu(G) \leqslant 3$ if and only if G is planar.

Colin de Verdière conjectured that $\gamma(G) \leqslant \mu(G) + 1$, where $\gamma(G)$ is the coloring number of G. This conjecture would follow from Hadwiger's conjecture [as $\mu(K_n) = n - 1$] and is true for $\mu(G) \leqslant 4$.

Graph G is a *clique sum* of graphs G_1 and G_2 if $VG = VG_1 \cup VG_2$ and $EG = EG_1 \cup EG_2$, where $VG_1 \cap VG_2$ is a clique both in G_1 and in G_2. Note that for the coloring number γ one has that $\gamma(G) = \max\{\gamma(G_1), \gamma(G_2)\}$ if G is a clique sum of G_1 and G_2. A similar relation holds for the size of the largest clique minor in a graph.

We therefore are interested in studying the behavior of $\mu(G)$ under clique sums (cf. also [4]). A critical example is the graph $K_{t+3} \setminus \Delta$ (the graph obtained from the complete graph K_{t+3} by deleting the edges of a triangle). One has $\mu(K_{t+3} \setminus \Delta) = t + 1$ [since the star $K_4 \setminus \Delta$ has $\mu(K_4 \setminus \Delta) = 2$ and since adding a new vertex adjacent to all existing vertices increases μ by 1].

However, $K_{t+3} \setminus \Delta$ is a clique sum of K_{t+1} and $K_{t+2} \setminus e$ (the graph obtained from K_{t+2} by deleting an edge), with common clique of size t. Both K_{t+1} and $K_{t+2} \setminus e$ have $\mu = t$. So, generally one does not have that, for fixed t, the property $\mu(G) \leqslant t$ is maintained under clique sums. Similarly, $K_{t+3} \setminus \Delta$ is a clique sum of two copies of $K_{t+2} \setminus e$, with common clique of size $t + 1$.

These examples where μ increases by taking a clique sum are in a sense the only cases: We show that if G is a clique sum of G_1 and G_2, with common clique S, then $\mu(G) > t := \max\{\mu(G_1), \mu(G_2)\}$ if and only if $t > 0$ and either $|S| = t$ and $G - S$ has three components, the contraction of which makes with S a $K_{t+3} \setminus \Delta$, or $|S| = t + 1$ and $G - S$ has two components, the contraction of which makes with S a $K_{t+3} \setminus \Delta$. Moreover, if $\mu(G) > t$, then $\mu(G) = t + 1$ and $\mu(G_1) = \mu(G_2) = t$.

So $\mu(G) = \max\{\mu(G_1), \mu(G_2)\}$ if and only if G does not contain $K_{t+3} \setminus \Delta$ as a minor.

In Section 2 we give the definition of Colin de Verdière's invariant, including an alternative linear algebraic characterization of the "strong Arnold hypothesis." In Section 3 we prove a lemma, while in Section 4 we derive our main characterization.

If M is a matrix, then M_K denotes the submatrix of M induced by the row and column indices in K. Similarly, if x is a vector, then x_K denotes the subvector of x induced by the indices in K. We denote the ith eigenvalue (from below) of M by $\lambda_i(M)$.

2. COLIN DE VERDIÈRE'S INVARIANT

We describe Colin de Verdière's invariant. Important is a certain general position assumption for matrices called the strong Arnold hypothesis. We here formulate it and give an equivalent linear algebraic characterization.

Let $M = (m_{i,j})$ be a symmetric $n \times n$ matrix. Let $R(M)$ be the set of all symmetric $n \times n$ matrices A with $\operatorname{rank}(A) = \operatorname{rank}(M)$. Let $S(M)$ be the set of all symmetric $n \times n$ matrices $A = (a_{i,j})$ such that $a_{i,j} = 0$ whenever $i \neq j$ and $m_{i,j} = 0$.

The matrix M is said to fulfill the *strong Arnold hypothesis* (*SAH*) if $R(M)$ intersects $S(M)$ at M "transversally"; that is, if the tangent space of $R(M)$ at M and the tangent space of $S(M)$ at M together span the space of all symmetric $n \times n$ matrices. In other words, if the intersection of the normal spaces at M of $R(M)$ and $S(M)$ only consists of the all-zero matrix.

The tangent space of $R(M)$ at M consists of all symmetric $n \times n$ matrices N such that $x^T N x = 0$ for each $x \in \ker(M)$. Thus the normal space of $R(M)$ at M is equal to the space generated by all matrices xx^T with $x \in \ker(M)$. This space is equal to the space of all symmetric $n \times n$ matrices X satisfying $MX = 0$. Trivially, the normal space of $S(M)$ at M consists of all symmetric $n \times n$ matrices $X = (x_{i,j})$ such that $x_{i,j} = 0$ whenever $i = j$ or $m_{i,j} \neq 0$. Therefore, the SAH is equivalent to:

> there is no nonzero symmetric $n \times n$ matrix $X = (x_{i,j})$ such that $MX = 0$ and such that $x_{i,j} = 0$ whenever $i = j$ or $m_{i,j} \neq 0$. (1)

Now Colin de Verdière's invariant $\mu(G)$ is defined as follows. Let G be an undirected graph, which throughout this paper we assume without loss of generality to have vertex set $\{1, \dots, n\}$. Then $\mu(G)$ is the largest corank of any symmetric $n \times n$ matrix $M = (m_{i,j})$ satisfying:

> M has exactly one negative eigenvalue (of multiplicity 1), and for all i, j with $i \neq j$, $m_{i,j} < 0$ if i and j are adjacent, and $m_{i,j} = 0$ otherwise, (2)

and such that M fulfills the SAH. [The *corank* corank(M) of a matrix M is the dimension of its kernel.]

It turns out, as proved in [2], that if G' is a minor of G, then $\mu(G') \leqslant \mu(G)$. (In proving this, the SAH is essential.) So for each fixed t, the class of graphs G satisfying $\mu(G) \leqslant t$ is closed under taking minors. Hence, by the theorem of Robertson and Seymour [6] there is a finite collection of "forbidden minors" for such a class of graphs.

Colin de Verdière [2] showed that the graphs G satisfying $\mu(G) \leqslant 1$ are exactly the paths, those satisfying $\mu(G) \leqslant 2$ are exactly the outerplanar graphs, and those satisfying $\mu(G) \leqslant 3$ are exactly the planar graphs. If $\mu(G) \leqslant 4$, then G is linklessly embeddable, since each graph G in the complete class of forbidden minors found by Robertson, Seymour, and Thomas [7] has $\mu(G) > 4$ (cf. Bacher and Colin de Verdière [1]). In fact, Robertson, Seymour, and Thomas [8] conjecture that also the reverse implication holds.

3. A LEMMA

The following lemma gives us some tools:

LEMMA. *Let $G = (V, E)$ be a graph and let M be a matrix satisfying (2). Let $S \subseteq V$ and let $C_1, \ldots, C_m$ be the components of $G - S$. Then*:

(i) *If $\lambda_1(M_{C_1}) < 0$, then $\lambda_1(M_{C_j}) > 0$ for all $j \neq 1$.*

(ii) *If $\lambda_1(M_{C_1}) = 0$, then there are at least* corank$(M) - |S| + 2$ *components C_i with $\lambda_1(M_{C_i}) = 0$.*

(iii) *If M fulfills the SAH, then there are at most three components C_i with $\lambda_1(M_{C_i}) = 0$.*

Proof. If (i) does not hold, we may assume that $\lambda_1(M_{C_1}) < 0$ and $\lambda_1(M_{C_2}) \leqslant 0$. Let z, x_1, and x_2 be the eigenvectors belonging to the smallest eigenvalues of M, M_{C_1}, and M_{C_2}, respectively. By the Perron-Frobenius theorem we may assume that $z, x_1, x_2 > 0$ and by scaling that $z_{C_1}^T x_1 = z_{C_2}^T x_2$.

Define $y \in \mathbb{R}^n$ by $y_i := (x_1)_i$ if $i \in C_1$, $y_i := -(x_2)_i$ if $i \in C_2$, and $y_i := 0$ if $i \notin C_1 \cup C_2$. Then $z^T y = z_{C_1}^T x_1 - z_{C_2}^T x_2 = 0$ and $y^T M y = x_1^T M_{C_1} x_1 + x_2^T M_{C_2} x_2 < 0$. However, $z^T y = 0$ and $y^T M y < 0$ imply that $\lambda_2(M) < 0$, contradicting (2).

To see (ii), if $\lambda_1(M_{C_1}) = 0$, then by (i), $\lambda_1(M_{C_i}) \geqslant 0$ for all i; that is, M_{C_i} is positive semidefinite for each i. Let D be the vector space of all vectors $x \in \ker(M)$ with $x_s = 0$ for all $s \in S$. Then:

for each vector $x \in D$ and each component C_i of $G - S$, $x_{C_i} = 0$, $x_{C_i} > 0$ or $x_{C_i} < 0$; if moreover $\lambda_1(M_{C_i}) > 0$, then $x_{C_i} = 0$. (3)

Indeed, if $x \in D$, then $M_{C_i} x_{C_i} = 0$. Hence if $x_{C_i} \neq 0$ (as M_{C_i} is positive semidefinite), $\lambda_1(M_{C_i}) = 0$ and x_{C_i} is an eigenvector belonging to $\lambda_1(M_{C_i})$, and hence (by the Perron-Frobenius theorem) $x_{C_i} > 0$ or $x_{C_i} < 0$.

Let m' be the number of components C_i with $\lambda_1(M_{C_i}) = 0$. By (3), $\dim(D) \leqslant m' - 1$ (since each nonzero $x \in D$ has both positive and negative components, as it is orthogonal to z).

Since $\lambda_1(M_{C_1}) = 0$, there exists a vector $w > 0$ such that $M_{C_1} w = 0$. Let F be the vector space of all vectors x_S with $x \in \ker(M)$. Suppose that $\dim(F) = |S|$. Let j be a vertex in S adjacent to C_1. Then there is a vector $y \in \ker(M)$ with $y_j = -1$ and $y_i = 0$ if $i \in S \setminus \{j\}$. Let u be the jth column of M. So $u_{C_1} = M_{C_1} y_{C_1}$. Since $u_{C_1} \leqslant 0$ and $u_{C_1} \neq 0$, we have $0 > u_{C_1}^T w = y_{C_1}^T M_{C_1} w = 0$, a contradiction.

Hence $\dim(F) \leqslant |S| - 1$, and so

$$m' - 1 \geqslant \dim(D) = \operatorname{corank}(M) - \dim(F) \geqslant \operatorname{corank}(M) - |S| + 1. \tag{4}$$

If (iii) does not hold, we may assume that $\lambda_1(M_{C_i}) = 0$, for $i = 1, \ldots, 4$. Let x_i be an eigenvector belonging to the smallest eigenvalue of M_{C_i}, for $i = 1, \ldots, 4$. Let z be the eigenvector belonging to smallest eigenvalue of M. We may assume that $z, x_1, \ldots, x_4 > 0$ and that $z_{C_1}^T x_1 = z_{C_2}^T x_2$ and $z_{C_3}^T x_3 = z_{C_4}^T x_4$. Define the vectors y_1 and y_2 by $(y_1)_i := (x_1)_i$ if $i \in C_1$, $(y_1)_i := -(x_2)_i$ if $i \in C_2$, and $(y_1)_i := 0$ if $i \notin C_1 \cup C_2$, and $(y_2)_i := (x_3)_i$ if $i \in C_3$, $(y_2)_i := -(x_4)_i$ if $i \in C_4$, and $(y_2)_i := 0$ if $i \notin C_3 \cup C_4$. Then $z^T y_1 = z_{C_1}^T x_1 - z_{C_2}^T x_2 = 0$ and $z^T y_2 = z_{C_3}^T x_3 - z_{C_4}^T x_4 = 0$. Since $y_1^T M y_1 = x_1^T M_{C_1} x_1 + x_2^T M_{C_2} x_2 = 0$ and similarly $y_2^T M y_2 = 0$, the vectors y_1 and y_2 belong to $\ker(M)$.

Define $X := y_1 y_2^T + y_2 y_1^T$. Then $x_{i,j} \neq 0$ implies $i \in C_1 \cup C_2$ and $j \in C_3 \cup C_4$ or conversely. As $MX = 0$, this contradicts the SAH. ∎

4. CLIQUE SUMS OF GRAPHS

Now let G be a clique sum of G_1 and G_2. Let $S := VG_1 \cap VG_2$ and $t := \max\{\mu(G_1), \mu(G_2)\}$. For any $U \subseteq VG$, let $N(U)$ denote the set of vertices in $VG \setminus U$ that are adjacent to at least one vertex in U.

THEOREM. *If $\mu(G) > t$, then $\mu(G) = t + 1$ and we can contract two or three components of $G - S$ so that the contracted vertices together with S form a $K_{t+3} \setminus \Delta$.*

Proof. We apply induction on $|VG| + |S|$. Let M be a matrix satisfying (2) and fulfilling the SAH, with corank equal to $\mu(G)$. We first show that $\lambda_1(M_C) \geqslant 0$ for each component C of $G - S$. Suppose $\lambda_1(M_C) < 0$. Hence by (i) of the lemma, $\lambda_1(M_{C'}) > 0$ for each other component C'. Let G' be the subgraph of G induced by $C \cup S$; so G' is a subgraph of G_1 or G_2. Let L be the union of the other components, so $\lambda_1(M_L) > 0$. We write

$$M = \begin{pmatrix} M_C & U_C & 0 \\ U_C^T & \Pi & U_L \\ 0 & U_L^T & M_L \end{pmatrix}. \tag{5}$$

Let

$$A := \begin{pmatrix} I & 0 & 0 \\ 0 & I & -U_L M_L^{-1} \\ 0 & 0 & I \end{pmatrix}. \tag{6}$$

Then by Sylvester's law of inertia (cf. [5, Section 5.5]), the spectrum of the matrix

$$AMA^T = \begin{pmatrix} M_C & U_C & 0 \\ U_C^T & \Pi - U_L M_L^{-1} U_L^T & 0 \\ 0 & 0 & M_L \end{pmatrix} \tag{7}$$

has the same signature as the spectrum of M; that is, AMA^T has exactly one negative eigenvalue and has the same corank as M. Let $\Pi' = \Pi - U_L M_L^{-1} U_L^T$.

As M_L is positive definite, the matrix

$$M' := \begin{pmatrix} M_C & U_C \\ U_C^T & \Pi \end{pmatrix} \tag{8}$$

has exactly one negative eigenvalue and has the same corank as M. Since $(M_L)_{i,j} \leqslant 0$ if $i \neq j$, we know that $(M_L^{-1})_{i,j} \geqslant 0$ for all i, j. [Indeed, for any symmetric positive-definite matrix D, if each off-diagonal entry of D is nonpositive, then each entry of D^{-1} is nonnegative. This can be seen directly, and also follows from the theory of "M-matrices" (cf. [5, Section 15.2]): Without loss of generality, each diagonal entry of D is at most 1. Let $B := I - D$. So $B \geqslant 0$ and the largest eigenvalue of B is equal to $1 - \lambda_1(D) < 1$. Hence $D^{-1} = I + B + B^2 + B^3 + \cdots \geqslant 0$ (cf. Theorem 2 in Section 15.2 of [5]).]

Hence, $(\Pi')_{i,j} \leqslant 0$ for each i and j with $i \neq j$. Thus M' satisfies (2) with respect to G'.

The matrix M' also fulfills the SAH. To see this, let X' be a symmetric matrix with $M'X' = 0$ and $(X')_{ij} = 0$ if i and j are adjacent or if $i = j$. As S is a clique, we can write

$$X' = \begin{pmatrix} X_C' & Y \\ Y^T & 0 \end{pmatrix}. \tag{9}$$

Let $Z := -YU_L M_L^{-1}$ and

$$X := \begin{pmatrix} X_C' & Y & Z \\ Y^T & 0 & 0 \\ Z^T & 0 & 0 \end{pmatrix}. \tag{10}$$

Then X is a symmetric matrix with $(X)_{i,j} = 0$ if i and j are adjacent or if $i = j$, and $MX = 0$. So $X = 0$ and hence $X' = 0$.

It follows that $\mu(G') \geqslant \text{corank}(M') = \text{corank}(M) = \mu(G) > t$, a contradiction, since G' is a subgraph of G_1 or G_2.

So we have that $\lambda_1(M_C) \geqslant 0$ for each component C of $G - S$. Suppose next that $N(C) \neq S$ for some component C of $G - S$.

Assume that $C \subseteq VG_1$. Let H_1 be the graph induced by $C \cup N(C)$ and let H_2 be the graph induced by the union of all other components and S. So

G is also a clique sum of H_1 and H_2, with common clique $S' := N(C)$, and H_2 is a clique sum of $G_1 - C$ and G_2.

If $\mu(G) = \mu(H_2)$, then $\mu(H_2) > t' := \max\{\mu(G_1 - C), \mu(G_2)\}$. As $|VH_2| + |S| < |VG| + |S|$, by induction we know that $\mu(H_2) = t' + 1$, and thus $\mu(G) = \mu(H_2) = t' + 1 \leqslant t + 1$. Thus $t' = t$ and $\mu(G) = t + 1$. Moreover, either $|S| = t + 1$ and $H_2 - S$ has two components C', C'' with $N(C') = N(C'')$ and $|N(C')| = t$, or $|S| = t$ and $H_2 - S$ has three components C' with $N(C') = S$, and the theorem follows.

If $\mu(G) > \mu(H_2)$, then $\mu(G) > t' := \max\{\mu(H_1), \mu(H_2)\}$. As $|VG| + |S'| < |VG| + |S|$, we know that $\mu(G) = t' + 1$, implying $t' \geqslant t$, and that either $|S'| = t' + 1$ or $|S'| = t'$. However, $|S'| < |S| \leqslant t + 1 \leqslant t' + 1$, so $|S'| = t'$ and $t' = t$. Moreover, $G - S'$ has three components C' with $N(C') = S'$. This implies that $G - S$ has two components C' with $N(C') = S'$, and the theorem follows.

So we may assume that $N(C) = S$ for each component C. If $|S| > t$, then G_1 would contain a K_{t+2} minor, contradicting the fact that $\mu(G_1) \leqslant t$. So $|S| \leqslant t$. Since $\operatorname{corank}(M) > |S|$, we have $\lambda_1(M_C) = 0$ for at least one component C of $G - S$. Hence, by (ii) of the lemma, $G - S$ has at least $\operatorname{corank}(M) - |S| + 2 = \mu(G) - |S| + 2 \geqslant 3$ components C with $\lambda_1(M_C) = 0$, and by (iii) of the lemma, $\mu(G) - |S| + 2 \leqslant 3$, that is, $t \geqslant |S| \geqslant \mu(G) - 1 \geqslant t$. ■

We give as direct consequences the following corollaries:

COROLLARY 1. *Let G be a clique sum of G_1 and G_2 and let $S := VG_1 \cap VG_2$. Then $\mu(G) = \max\{\mu(G_1), \mu(G_2)\}$ if $\mu(G_1) \neq \mu(G_2)$, or $|S| < \mu(G_1)$, or $|S| = \mu(G_1)$ and $G - S$ has at most two components C with $N(C) = S$.*

COROLLARY 2. *Let G be a clique sum of G_1 and G_2 and let $t = \max\{\mu(G_1), \mu(G_2)\}$. Then $\mu(G) = t$ if and only if G does not have a $K_{t+3} \setminus \Delta$-minor.*

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